Almost sure convergence of weighted partial sums

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Abstract

In the paper sufficient conditions of covariance type are presented for weighted averages of random variables with arbitrary dependence structure to converge to 0, both for logarithmic and general weighting. As an application, an a.s. local limit theorem of Csáki, Földes and Révész is revisited and slightly improved.

1 Introduction

In the last decade many interesting extensions of classical limit theorems have been obtained as contributions to the so-called almost sure central limit theory. The first basic results were discovered independently by Brosamler [2] and Schatte [6], and slightly later by Lacey and Philipp [4].

Theorem A. (Brosamler, Schatte, Lacey, Philipp) Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$ and set $S_n = X_1 + \dots + X_n$. Then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\, \mathbf{I}\Big\{ \frac{S_k}{\sqrt{k}} \le x \Big\} = \Phi(x) \quad a.s. \tag{1}$$

for every real x, where $\Phi(x)$ is the standard normal distribution function, and $\mathbf{I}\{\cdot\}$ stands for the indicator of the event in curly brackets.

This result has been extended and generalized in several ways. In their 1991 paper [1] Berkes and Dehling provide a systematic study of logarithmic analogues of classical limit theorems. They also present an effective method which can be applied to all similar problems. It is based on the observation that, under very mild conditions, the a.s. limit behaviour of the sequences

$$\frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i}\, \mathbf{I}\Big\{ \frac{S_i - b_i}{a_i} < x \Big\} \quad\text{and}\quad \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i}\, \mathbf{P}\Big(\frac{S_i - b_i}{a_i} < x \Big)$$

coincide. More precisely, defining $\xi_i = \mathbf{I}\Big\{ \frac{S_i - b_i}{a_i} < x \Big\} - \mathbf{P}\Big(\frac{S_i - b_i}{a_i} < x \Big)$ we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i}\, \xi_i = 0 \quad a.s. \tag{2}$$

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Móri [5] found conditions that are sufficient for a more general sequence $\{\xi_i\}$ to guarantee (2). The main assumption of this result was the existence of a suitable function h as an upper bound of the covariances, in the form of $\mathbf{E}\xi_i\xi_j \le h(j/i)$, $1 \le i \le j$.

More precisely, let us define $l(x) = \log x$ for $e \le x$ and $l(x) = 1$ for $0 < x \le e$. Let $l_1(x) = l(x)$ and $l_k(x) = l(l_{k-1}(x))$ for $k \ge 2$.

Theorem B. (Móri [5]) Let $\xi_1, \xi_2, \dots$ be arbitrary random variables with finite variances. Suppose there exists a positive non-increasing function h on the positive numbers and a positive integer m such that

$$\int_1^\infty h(z)\frac{l_m(z)}{zl(z)}\,\mathrm{d}z < \infty, \tag{3}$$

and for any $1 \le i \le j$

$$\mathbf{E}(\xi_i\xi_j) \le h\Big(\frac{j}{i}\Big)\ . \tag{4}$$

Then

$$\lim_{n\to\infty} \frac{1}{l(n)} \sum_{i=1}^n \frac{1}{i}\,\xi_i = 0 \quad \text{a.s.}$$

In the present paper we derive a result of similar kind, but with an h which is not essentially bounded in the neighborhood of 1 (Theorem 1). This property of h enables us to apply our result for proving a.s. local limit theorems. As an example, in Section 4 we revisit a theorem of Csáki, Földes and Révész on the limit behaviour of the logarithmic averages

$$\frac{1}{\log n} \sum_{k=1}^n \frac{\mathbf{I}\{a_k \le S_k \le b_k\}}{k\,\mathbf{P}(a_k \le S_k \le b_k)}\ ,$$

where $b_k - a_k = o(\sqrt{k})$ is allowed. By using Theorem 1 we are able to prove the same result under somewhat weaker conditions.

Dealing with logarithmic averages one can naturally ask what could be said when the weighting is non-logarithmic; that is, when the asymptotic behaviour of the more general weighted averages

$$\frac{1}{\mathbf{B}(n)} \sum_{i=1}^n w_i\xi_i$$

is considered, where $\{w_k\}$ is a non-negative weighting sequence and $\mathbf{B}(n) = \sum_{k=1}^n w_k$. By applying a simple transformation we can adapt Theorem 1 to this case as well, thus obtaining a fairly general result on the stability of weighted averages of random variables with arbitrary dependence structure. That kind of problems will be studied in Section 3.

2 Logarithmic weighting

Dropping the boundedness of second moments we have to add a further condition on the bounding function h in the neighborhood of 1, and another one that regulates the growth of the second moments.

Theorem 1. *Let $\xi_1, \xi_2, \dots$ be arbitrary random variables with finite variances. Suppose there exists a non-increasing positive function h defined on $(1, \infty)$, such that for all $1 \le i < j$*

$$\mathbf{E}(\xi_i \xi_j) \le h\Big(\frac{j}{i}\Big), \tag{5}$$

Suppose that function h satisfies

$$\int_1^\infty h(z) \frac{l_m(z)}{z l(z)} \, \mathrm{d}z < \infty , \tag{6}$$

for some positive integer m; and

$$\int_0^1 h(z+1) \Big(\log \frac{1}{z}\Big)^{2(1+\varepsilon)} \mathrm{d}z < \infty , \tag{7}$$

for some $\varepsilon > 0$. Finally, suppose that for all $1 \le i$

$$\mathbf{E}(\xi_i^2) \le S(i) , \tag{8}$$

where the sequence $\{S\}$ satisfies

$$\sum_{i=1}^\infty \frac{1}{i^2} \, S(i) < \infty . \tag{9}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{l(n)} \sum_{i=1}^n \frac{1}{i} \, \xi_i = 0 \quad a.s.$$

Proof. For the sake of brevity let us introduce some notations. Let

$$\eta(t) = \sum_{1 \le i < t} \frac{1}{i} \, \xi_i,$$

where t is an arbitrary positive number, greater than 1. For positive real numbers $1 \le s < t$ and a positive increasing sequence $\{a_k\}$ we shall adopt the abbreviations

$$\nu(s,t \mid \{a_k\}) = \#\{k : s \le a_k < t\},$$
$$\mu(s,t \mid \{a_k\}) = \max_{s \le a_k < t} |\eta(a_k) - \eta(s)| .$$

We are going to apply the method of subsequences in the same way as it was done in the proof of Theorem B in [5]. We will prove $\eta(n)/l(n) \to 0$ along more and more dense subsequences of positive integers, arriving at $\mathbb{N}$ itself in the end.

We first show that

$$\mathbf{E}\left(\sum_{k=1}^{\infty} \frac{\eta^2(N_k)}{l^2(N_k)}\right) < \infty \tag{10}$$

for a sufficiently sparse subsequence $\{N_k\}$. From this one can conclude that

$$\lim_{k\to\infty} \eta(N_k)/l(N_k) = 0 \quad \text{a.s.}$$

Then the following iterative steps will be carried out repeatedly. At each step we consider a subsequence $\{a_k\}$ of the positive integers, and a sub-subsequence $\{b_k\} \subset \{a_k\}$. We start from the limit relation $\lim_{k\to\infty} \eta(b_k)/l(b_k) = 0$ a.s., and aim at the same with a_k in place of b_k, by checking if

$$\lim_{k\to\infty} \mu\big(b_k, b_{k+1} \mid \{a_n\}\big)/l(b_k) = 0 \quad \text{a.s.} \tag{11}$$

It is clearly sufficient, since for $b_k \le a_n < b_{k+1}$ we have

$$\frac{|\eta(a_n)|}{l(a_n)} \le \frac{|\eta(b_k)| + \mu\big(b_k, b_{k+1} \mid \{a_n\}\big)}{l(b_k)}.$$

We will prove (11) by showing the finiteness of the expectation

$$\mathbf{E}\left(\sum_{k=1}^{\infty} \frac{\mu^2\big(b_k, b_{k+1} \mid \{a_n\}\big)}{l^2(b_k)}\right). \tag{12}$$

At the first step $b_n = N_n$ is taken. At every subsequent step, for $\{b_n\}$ we choose the $\{a_n\}$ of the preceding step. We finally complete the proof by setting $\{a_n\} = \mathbb{N}$ at the last step.

In order to show the finiteness of (10) and (12) let us define the function

$$g(s,t) = 2 \sum_{s \le i < j-1 < t-1} \frac{1}{ij}\, h\Big(\frac{j}{i}\Big) + 3 \sum_{s \le i < t} \frac{1}{i^2}\, S(i).$$

By applying the estimation $\mathbf{E}|\xi_i\xi_{i+1}| \le \big(\mathbf{E}\xi_i^2\, \mathbf{E}\xi_{i+1}^2\big)^{1/2} \le \frac{1}{2}\big(\mathbf{E}\xi_i^2 + \mathbf{E}\xi_{i+1}^2\big)$ we clearly have

$$\mathbf{E}(\eta(t) - \eta(s))^2 = \sum_{s \le i < t}\ \sum_{s \le j < t} \frac{1}{ij}\, \mathbf{E}(\xi_i\xi_j) \le g(s,t)\ . \tag{13}$$

In addition, for any $1 \le s < t < u$ the subadditive property

$$g(s,t) + g(t,u) \le g(s,u)\,.$$

holds. So we can apply Serfling's maximal inequality [7] to the sequence $\{\xi_i/i\}$:

$$\mathbf{E}\mu^2\big(s,t \mid \{a_k\}\big) \le 6\, l\big(\nu(s,t \mid \{a_k\})\big)^2 g(s,t). \tag{14}$$

Let us estimate $g(s,t)$. For arbitrary $1 \le i < j$ let D_{ij} denote the parallelogram with vertices $(i,j), (i+1,j), (i+1,j+1), (i+2,j+1)$. Under the assumption of $(x,y) \in D_{ij}$, $i \ne j$ we have

$$\frac{1}{xy}\, h\Big(\frac{y}{x}\Big) \ge \frac{1}{(i+2)(j+1)}\, h\Big(\frac{j}{i}\Big) \ge \frac{1}{6ij}\, h\Big(\frac{j}{i}\Big),$$

hence

$$\begin{aligned} g(s,t) &\le \sum_{s\le i,\, i+2\le j<t} 12 \int_{D_{ij}} \frac{1}{xy}\, h\Big(\frac{y}{x}\Big)\, \mathrm{d}x\, \mathrm{d}y + 3 \sum_{s\le i<t} \frac{1}{i^2} S(i) \\ &\le 12 \int_{\{s\le x\le y-1<t\}} \frac{1}{xy}\, h\Big(\frac{y}{x}\Big)\, \mathrm{d}x\, \mathrm{d}y + 3 \sum_{s\le i<t} \frac{1}{i^2}\, S(i)\ . \end{aligned}$$

Extending the range of integration, then substituting $z = y/x$ we get

$$\int_{\{s\le x\le y-1<t\}} \frac{1}{xy}\, h\left(\frac{y}{x}\right) \mathrm{d}x\, \mathrm{d}y \le \int_{\left\{s\le x<t,\ 1\le \frac{y}{x} < \frac{t+1}{s}\right\}} \frac{1}{xy}\, h\left(\frac{y}{x}\right) \mathrm{d}x\, \mathrm{d}y = \int_s^t \frac{\mathrm{d}x}{x} \int_1^{\frac{t+1}{s}} \frac{h(z)}{z}\, \mathrm{d}z.$$

The first integral on the right-hand side is equal to $\log(t/s)$. In the second one one can use that the integrand is a decreasing function of z, hence

$$\int_1^{\frac{t+1}{s}} \frac{h(z)}{z}\, \mathrm{d}z \le \frac{t+1}{t} \int_1^{\frac{t}{s}} \frac{h(z)}{z}\, \mathrm{d}z \le 2 \int_1^{\frac{t}{s}} \frac{h(z)}{z}.$$

From all these one can conclude that

$$g(s,t) \le 24 \log\Big(\frac{t}{s}\Big) \int_1^{\frac{t}{s}} \frac{h(z)}{z}\, \mathrm{d}z + 3 \sum_{s\le i<t} \frac{1}{i^2}\, S(i)\ . \tag{15}$$

Particularly, from (13), (15), and assumption (9) it follows that

$$\begin{aligned} \mathbf{E}\left(\sum_{k=1}^{\infty} \frac{\eta^2(N_k)}{l^2(N_k)}\right) &\le \sum_{k=1}^{\infty} \frac{g(1,N_k)}{l^2(N_k)} \\ &\le C \sum_{k=1}^{\infty} \frac{1}{l(N_k)} \int_1^{N_k} \frac{h(z)}{z}\, \mathrm{d}z + C \sum_{k=1}^{\infty} \frac{1}{l^2(N_k)} \sum_{1\le i<N_{k+1}} \frac{1}{i^2}\, S(i) \\ &\le C \int_1^{\infty} \frac{h(z)}{z} \Big(\sum_{N_k>z} \frac{1}{l(N_k)}\Big)\, \mathrm{d}z + C \sum_{k=1}^{\infty} \frac{1}{l^2(N_k)}. \end{aligned} \tag{16}$$

if $N_k \to \infty$. Here and in the sequel C always stands for a suitable positive constant that may change at every appearance.

Let us turn to the estimation of (12). Let us briefly denote $\dfrac{l^2\big(\nu(b_k, b_{k+1} \mid \{a_k\})\big)}{l^2(b_k)}$ by γ_k. Then by (14), (15) and assumption (9) we get

$$\begin{aligned}
(17) \qquad \mathbf{E}\left(\sum_{k=1}^{\infty} \frac{\mu^2\big(b_k, b_{k+1} \mid \{a_n\}\big)}{l^2(b_k)}\right) &\le C \sum_{k=1}^{\infty} \gamma_k \, g(b_k, b_{k+1}) \\
&\le C \sum_{k=1}^{\infty} \gamma_k \log\Big(\frac{b_{k+1}}{b_k}\Big) \int_1^{\frac{b_{k+1}}{b_k}} \frac{h(z)}{z} \,\mathrm{d}z + C \sum_{k=1}^{\infty} \gamma_k \sum_{b_k \le i < b_{k+1}} \frac{1}{i^2}\, S(i) \\
&\le C \int_1^{\infty} \frac{h(z)}{z} \Bigg[\sum_{\frac{b_{k+1}}{b_k} > z} \gamma_k \log\Big(\frac{b_{k+1}}{b_k}\Big)\Bigg] \mathrm{d}z + C \sup_{k \ge 1} \gamma_k \,.
\end{aligned}$$

Now we are ready to realize our plan of work. For arbitrary positive integer r let us define $N_k^{(r)} = \Big\lceil \exp\Big(\exp\Big(\frac{k}{l_r^2(k)}\Big)\Big)\Big\rceil$. Set $N_k = N_k^{(m)}$. In order to prove (10) we use estimation (16). Routine calculations show that

$$(18) \qquad \sum_{N_k^{(r)} > z} \frac{1}{l(N_k^{(r)})} \le C \, \frac{l_{r+2}^2(z)}{l(z)} \,,$$

hence by (6) the first term in the last row of (16) is finite. On the other hand, the finiteness of the second sum is obvious. Thus (10) holds.

We continue the proof with the first $m-1$ steps. At step i let us choose $a_k = N_k^{(m-i)}$ and $b_k = N_k^{(m+1-i)}$. It is easy to see that with the above defined $N_k^{(r)}$ the following asymptotic statements are valid.

$$(19) \qquad \nu\Big(N_k^{(r)}, N_{k+1}^{(r)} \mid \{N_n^{(r-1)}\}\Big) \sim \frac{l_{r-1}^2(k)}{l_r^2(k)}, \quad \text{and} \quad l\left(\frac{N_{k+1}^{(r)}}{N_k^{(r)}}\right) \sim \frac{l(N_k^{(r)})}{l_r^2(k)} \,.$$

Hence at step $i = m+1-r$ we have

$$(20) \qquad \gamma_k \le C \, \frac{l_r^2(k)}{l^2(N_k^{(r)})}, \quad \text{and} \quad \gamma_k \log\Big(\frac{b_{k+1}}{b_k}\Big) \le C \, \frac{1}{l(N_k^{(r)})}.$$

Note that γ_k is bounded. Furthermore, condition $b_{k+1}/b_k > z$ implies that $l(N_k^{(r)}) > C\, l_t^2(k) l(z)$ and $k > l_2(z)$, therefore $N_k^{(r)} > \exp\big(C\, l_{r+2}^2(z) l(z)\big)$. Hence by (18) and (20) we obtain that

$$(21) \qquad \sum_{\frac{b_{k+1}}{b_k} > z} \gamma_k \log\Big(\frac{b_{k+1}}{b_k}\Big) \le C \, \frac{l_{r+2}^2\big(l_{r+2}^2(z) l(z)\big)}{l_{r+2}^2(z) l(z)} \le C \, \frac{1}{l(z)}.$$

This, combined with (6) and (17) proves the finiteness of (12). In the end we conclude that $\eta(N_k^{(1)})/l(N_k^{(1)}) \to 0$.

At the next step let $b_k = N_k^{(1)} = \lceil \exp\bigl(\exp\bigl(k/l^2(k)\bigr)\bigr)\rceil$ and $a_k = \lceil \exp\bigl(\exp\bigl(k^{1/3}\bigr)\bigr)\rceil$. Then clearly $\nu\bigl(b_k, b_{k+1} \bigm| \{a_n\}\bigr) = O(k^2)$; consequently, (20) is valid with $r = 1$. Hence (21) follows, completing the step in the same way as before.

Next, let $b_k = \lceil \exp\bigl(\exp\bigl(k^{1/3}\bigr)\bigr)\rceil$ and $a_k = \lceil \exp\bigl(k^{1+\varepsilon}\bigr)\rceil$, where ε is the constant introduced in assumption (7). Similarly to the preceding steps we obtain that $\nu\bigl(b_k, b_{k+1} \bigm| \{a_n\}\bigr) = O\bigl(\exp\bigl(\frac{1}{1+\varepsilon}k^{1/3}\bigr)\bigr)$ and $l(\frac{b_{k+1}}{b_k}) \sim \frac{1}{3}k^{-2/3}l(b_k)$. Thus we can see again that

$$\gamma_k \le C\,\frac{k^{2/3}}{l^2(b_k)} \quad \text{and} \quad \gamma_k \log\Bigl(\frac{b_{k+1}}{b_k}\Bigr) \le C\,\frac{1}{l(b_k)}. \tag{22}$$

Moreover, if $b_{k+1}/b_k > z$, then $l(b_k) > C\,l(z)k^{2/3} > C\,l(z)l_2^2(z)$; and one can readily verify that

$$\sum_{l(b_k) > z} \frac{1}{l(b_k)} \le C\,\frac{l^2(z)}{z}.$$

Hence estimation (21) is also valid here:

$$\sum_{\frac{b_{k+1}}{b_k} > z} \gamma_k \log\Bigl(\frac{b_{k+1}}{b_k}\Bigr) \le C\,\frac{1}{l(z)}.$$

We can finish the step by plugging back this and (22) into (17).

After all these calculations let $b_k = \lceil \exp(k^{1+\varepsilon})\rceil$, and $a_k = \lceil \exp(\log k)^{2+2\varepsilon}\rceil$. Then obviously $\nu\bigl(b_k, b_{k+1} \bigm| \{a_n\}\bigr) \le C\,\exp(\sqrt{k})$, $\log\bigl(b_{k+1}/b_k\bigr) \le C\,k^{\varepsilon}$, hence $\gamma_k \le C\,k^{-(1+2\varepsilon)}$, and

$$\sum_{\frac{b_{k+1}}{b_k} > z} \gamma_k \log\Bigl(\frac{b_{k+1}}{b_k}\Bigr) \le C \sum_{k^{\varepsilon} > C l(z)} \frac{1}{k^{1+\varepsilon}} \le C\,\frac{1}{l(z)}.$$

As before, these are already sufficient to complete the step.

Finally, let $b_k = \lceil \exp(\log k)^{2+2\varepsilon}\rceil$, and $a_k = k$. Then $\nu\bigl(b_k, b_{k+1} \bigm| \{a_n\}\bigr) \le C\,b_k$, $\log\bigl(b_{k+1}/b_k\bigr) \sim 2(1+\varepsilon)k^{-1}(\log k)^{1+2\varepsilon}$, hence $\gamma_k = O(1)$ (and it is not hard to see that nothing better can be said). This time the last line of (17) cannot be used, because the infinite sum of $\gamma_k \log\bigl(b_{k+1}/b_k\bigr)$ is divergent. Looking at the middle line we can see that the second term is finite, and in the first term $h(z)/z$ is only integrated in the neighbourhood of 1, thus this is the point where assumption (7) might be of use.

Let us denote $\log\bigl(b_{k+1}/b_k\bigr)$ by w_k, then the sequence w_k is eventually decreasing. Since $w_k \sim \dfrac{b_{k+1}}{b_k} - 1$, by the decreasing property of h we get

$$\int_1^{\frac{b_{k+1}}{b_k}} \frac{h(z)}{z}\,\mathrm{d}z \le C \int_0^{w_k} h(z+1)\,\mathrm{d}z\,.$$

Thus the first series in the middle line of (17) can be treated in the following way.

$$\sum_{k=1}^{\infty}\gamma_k \log\Big(\frac{b_{k+1}}{b_k}\Big)\int_1^{\frac{b_{k+1}}{b_k}}\frac{h(z)}{z}\,\mathrm{d}z \le C\sum_{k=1}^{\infty} w_k \int_0^{w_k} h(z+1)\,\mathrm{d}z = C\sum_{k=1}^{\infty}\int_{w_{k+1}}^{w_k} h(z+1)\Big(\sum_{i=1}^{k} w_i\Big)\,\mathrm{d}z$$

$$\le C\sum_{k=1}^{\infty}\int_{w_{k+1}}^{w_k} h(z+1)(\log k)^{2(1+\varepsilon)}\,\mathrm{d}z \le C\int_0^1 h(z+1)\Big(\log\frac{1}{z}\Big)^{2(1+\varepsilon)}\,\mathrm{d}z < \infty.$$

by assumption (7). Hence expectation (12) is finite, consequently $\lim_{n\to\infty}\eta(n)/l(n) = 0$ a.s., which was to be proved. □

3 General weighting

Throughout this section we consider the following model. Let $\xi_1, \xi_2, \ldots$ be a sequence of arbitrary random variables with finite variances, and $\{w_k\}$ an arbitrary sequence of positive weights with partial sums $B(n) = \sum_{k=1}^n w_k$.

In Theorem 2 below we are about to state conditions for the weighted sums $B(n)^{-1}\sum_{k=1}^n w_k\xi_k$ to converge to 0, similar to those in Theorem 1.

Theorem 2. *Suppose that*

(23) $$w_k \le (1-\varepsilon)B(k), \quad k > 1,$$

for some $\varepsilon > 0$, and

(24) $$\sum_{k=1}^{\infty} e^{-mB(k)} < \infty,$$

for some positive integer m. Suppose there exists a non-increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(25) $$\int_1^{\infty} h(z)\frac{l_m(z)}{z}\,\mathrm{d}z < \infty,$$

(26) $$\int_0^1 h(z)\Big(\log\frac{1}{z}\Big)^{2(1+\varepsilon)}\,\mathrm{d}z < \infty,$$

and, in addition,

(27) $$\mathbf{E}\big(\xi_i\xi_j\big) \le h\big(B(j) - B(i-1)\big)$$

for every $1 \le i \le j$. Then we have

(28) $$\lim_{n\to\infty}\frac{1}{B(n)}\sum_{k=1}^{n} w_k\xi_k = 0$$

with probability 1.

Proof. We begin the proof by separating the summands in (28) into two sums. Introducing the notations

$$V(n) = \frac{1}{B(n)} \sum_{k=1}^{n} w_k \mathbf{I}\left\{e^{-2mB(k)} < w_k\right\} \xi_k \ ,$$
$$W(n) = \frac{1}{B(n)} \sum_{k=1}^{n} w_k \mathbf{I}\left\{w_k \le e^{-2mB(k)}\right\} \xi_k \ ,$$

we show that both $V(n)$ and $W(n)$ converge to 0, with probability 1.

We first investigate the convergence of $V(n)$.

Let $M = 3m/\varepsilon$. For $e^{MB(k-1)} \le i < e^{MB(k)}$ define

$$k_i = k, \quad \widetilde{\xi}_i = \xi_k \mathbf{I}\left\{e^{-2mB(k)} < w_k\right\}, \quad \widetilde{S}(i) = h(w_k)\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\}.$$

Furthermore, let $\widetilde{h}(z) = h(M^{-1}\log z)$. We will apply Theorem 1 to the sequence $\{\widetilde{\xi}_i\}$. To this end we check conditions (5)–(9).

For all $1 \le i < j$ we can write

$$\mathbf{E}\big(\widetilde{\xi}_i\widetilde{\xi}_j\big) \le \mathbf{E}\left(\xi_{k_i}\xi_{k_j}\right) \le h\big(B(k_j) - B(k_i - 1)\big) \le \widetilde{h}\Big(\frac{j}{i}\Big) \ , \tag{29}$$

and for $1 \le i$ we have

$$\mathbf{E}\big(\widetilde{\xi}_i^2\big) = \mathbf{E}\xi_{k_i}^2 \mathbf{I}\left\{e^{-2mB(k_i)} < w_{k_i}\right\} \le h\big(B(k_i) - B(k_i - 1)\big)\mathbf{I}\left\{e^{-2mB(k_i)} < w_{k_i}\right\}$$
$$= h(w_{k_i})\mathbf{I}\left\{e^{-2mB(k_i)} < w_{k_i}\right\} = \widetilde{S}(i) \ .$$

Conditions (6) and (7) of Theorem 1 are fulfilled with $\widetilde{h}$ in place of h. Indeed, by substituting $z = M^{-1}l(x)$ and using (25) we have

$$\int_{e^M}^{\infty} \widetilde{h}(x)\frac{l_{m+1}(x)}{xl(x)}\,\mathrm{d}x = \int_{e^M}^{\infty} h\Big(\frac{\log x}{M}\Big)\frac{l_m(l(x))}{xl(x)}\,\mathrm{d}x = \int_{1}^{\infty} h(z)\frac{l_m(Mz)}{z}\,\mathrm{d}z < \infty \ ,$$

and from (26), by substituting $z = M^{-1}\log(x+1)$ we get

$$\int_0^1 \widetilde{h}(x+1)\Big(\log\frac{1}{x}\Big)^{2(1+\varepsilon)}\mathrm{d}x \le \int_0^1 h\Big(\frac{\log(x+1)}{M}\Big)\Big(\log\frac{1}{\log(x+1)}\Big)^{2(1+\varepsilon)}\mathrm{d}x$$
$$\le \int_0^{\frac{\log 2}{M}} h(z)\Big(\log\frac{1}{Mz}\Big)^{2(1+\varepsilon)} e^{Mz}\,\mathrm{d}z \le 2\int_0^1 h(z)\Big(\log\frac{1}{z}\Big)^{2(1+\varepsilon)}\mathrm{d}z < \infty.$$

Regarding $\widetilde{S}$ we first note that $B(k-1) > \varepsilon B(k)$ by (23). Hence we have

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{i^2}\,\widetilde{S}(i) &= \sum_{i=1}^{\infty} \frac{1}{i^2}\,h(w_{k_i})\mathbf{I}\left\{e^{-2mB(k_i)} < w_{k_i}\right\} \\ &= \sum_{k=1}^{\infty}\Bigg(\sum_{e^{MB(k-1)}\le i<e^{MB(k)}} \frac{1}{i^2}\Bigg)\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\} h(w_k) \\ &\le C\sum_{k=1}^{\infty} h(w_k)e^{-MB(k-1)}\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\} \\ &\le C\sum_{w_k\ge 1} h(w_k)e^{-3mB(k)} + C\sum_{w_k<1} w_k h(w_k)e^{-mB(k)}.\end{aligned}$$

Here $h(w_k) \le h(1)$ if $w_k \ge 1$, and for $w_k < 1$ we have $w_k h(w_k) \le \int_0^1 h(z)\,\mathrm{d}z < \infty$. Thus, by (26) and (24), (9) holds with $\widetilde{S}$ in place of S.

Now, by Theorem 1 one obtains that

$$\frac{1}{MB(n)}\sum_{i<e^{MB(n)}} \frac{1}{i}\,\widetilde{\xi}_i \to 0 \quad \text{a.s.} \tag{30}$$

Finally, we will prove that the difference between the left-hand side of (30) and $V(n)$ tends to 0 with probability 1. Clearly,

$$\left|\sum_{e^{MB(k-1)}\le i<e^{MB(k)}} \frac{1}{i} - \int_{e^{MB(k-1)}}^{e^{MB(k)}} \frac{1}{x}\,\mathrm{d}x\right| \le e^{-MB(k-1)} \le e^{-3mB(k)} \tag{31}$$

Hence we get

$$\begin{aligned}&\left|\frac{1}{M}\sum_{i<e^{MB(n)}} \frac{1}{i}\,\widetilde{\xi}_i - \sum_{k=1}^{n} w_k\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\}\xi_k\right| \\ &\qquad = \frac{1}{M}\left|\sum_{k=1}^{n}\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\}\xi_k\Bigg(\sum_{e^{MB(k-1)}\le i<e^{MB(k)}} \frac{1}{i} - \int_{e^{MB(k-1)}}^{e^{MB(k)}} \frac{1}{x}\,\mathrm{d}x\Bigg)\right| \\ &\qquad \le \frac{1}{M}\sum_{k=1}^{\infty}\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\}|\xi_k|e^{-3mB(k)}.\end{aligned}$$

This series converges with probability 1, because its expectation is finite:

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbf{E}|\xi_k|e^{-3mB(k)}\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\} &\le \sum_{k=1}^{\infty}\sqrt{h(w_k)}\,e^{-3mB(k)}\mathbf{I}\left\{e^{-2mB(k)} < w_k\right\} \\ &\le \sum_{w_k\ge 1}\sqrt{h(w_k)}\,e^{-3mB(k)} + \sum_{w_k<1}\sqrt{w_k h(w_k)}\,e^{-2mB(k)} \\ &\le \left(\sqrt{h(1)} + \left(\int_0^1 h(z)\,\mathrm{d}z\right)^{1/2}\right)\sum_{k=1}^{\infty} e^{-mB(k)} < \infty.\end{aligned}$$

Let us turn to $W(n)$. Obviously,

$$\mathbf{E}\left(\sum_{n=1}^{\infty} \max\left\{|W(k)| : 2^n \le B(k) < 2^{N+1}\right\}\right) < \infty \tag{32}$$

would imply the a.s. convergence of $W(n)$ to 0.

The left-hand side of (32) can be estimated in the following way.

$$\begin{aligned}
&\mathbf{E}\left(\sum_{n=1}^{\infty} \max\left\{|W(k)| : 2^n \le B(k) < 2^{n+1}\right\}\right) \\
&\qquad \le \mathbf{E}\left(\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k:\, B(k)<2^{n+1}} \mathbf{I}\left\{w_k \le e^{-2mB(k)}\right\} w_k|\xi_k|\right) \\
&\qquad \le \sum_{k=1}^{\infty} \mathbf{I}\left\{w_k \le e^{-2mB(k)}\right\} w_k \mathbf{E}|\xi_k| \sum_{n:\, B(k)\le 2^{n+1}} \frac{1}{2^n} \\
&\qquad \le C \sum_{w_k<1} \sqrt{w_k h(w_k)}\, e^{-mB(k)} \frac{1}{B(k)} \le C \sum_{k=1}^{\infty} e^{-mB(k)} < \infty
\end{aligned}$$

by (24). □

Remark 1. Conditions (23) and (24) are obviously satisfied in the important particular case of weights $w_k = k^{\lambda}$, $\lambda \ge -1$. Though condition (23) would allow weights that grow exponentially fast, the real area of application is the case where the weights are bounded. At the other end, condition (24) is hurt if $kw_k \to 0$.

Remark 2. Though Theorem 1 itself is not a particular case of Theorem 2, a very similar but slightly weaker result can be derived from that; namely, $S(i)$ is supposed to increase with i, and condition (9) is to be replaced by the stronger one

$$\sum_{i=1}^{\infty} \frac{(\log i)^{2(1+\varepsilon)}}{i^2} S(i) < \infty\,.$$

4 Application: an a.s. local limit theorem

In the rest of the paper we revisit and slightly improve an a.s. local limit thoerem due to Csáki, Földes and Révész [3]. Let $X_1, X_2, \ldots$ be i.i.d. random variables with 0 mean and finite third moment, and let $S_n = X_1 + \cdots + X_n$ denote their partial sums. Let a_k, b_k be (extended) real numbers, $-\infty \le a_k \le 0 \le b_k \le \infty$. Define $p_k = \mathbf{P}(a_k \le S_k \le b_k)$, and

$$\alpha_k = \begin{cases} \dfrac{\mathbf{I}\{a_k \le S_k \le b_k\}}{p_k}, & \text{if } p_k \ne 0, \\ 1 & \text{if } p_k = 0, \end{cases}\,.$$

Theorem 3. *Suppose*

$$\sum_{k:\,p_k\neq 0} \frac{l_2^2(k)}{p_k\, k^{3/2} \log k} < \infty \tag{33}$$

holds. Then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\alpha_k}{k} = 1 \ a.s.$$

Remark 3. In Theorem 2.5 in [3] the same conclusion was drawn from a somewhat stronger assumption, namely, that

$$\sum_{\substack{1\le k\le n\\ p_k\neq 0}} \frac{\log k}{k^{3/2} p_k} = O(\log n) \tag{34}$$

as $n\to\infty$. From this latter assumption (33) can be derived; moreover, (33) is even implied by the following, similar to (34), but weaker condition.

$$\sum_{\substack{1\le k\le n\\ p_k\neq 0}} \frac{l_2^3(k)}{k^{3/2} p_k} = O(\log n). \tag{35}$$

Indeed,

$$\sum_{p_k\neq 0} \frac{l_2^2(k)}{p_k\, k^{3/2}\log k} \le C \sum_{p_k\neq 0} \frac{l_2^3(k)}{k^{3/2} p_k} \sum_{n>k} \frac{1}{n\log^2 n\, l_2^2(n)}$$
$$= C \sum_{n=1}^{\infty} \frac{1}{n\log^2 n\, l_2^2(n)} \sum_{\substack{1\le k\le n\\ p_k\neq 0}} \frac{l_2^3(k)}{k^{3/2} p_k} \le C \sum_{n>1} \frac{1}{n\log n\, l_2^2(n)} < \infty\ ,$$

by (35).

For the proof we shall need the following lemma, which is just a slightly modified version of Lemma 2.9 in [3].

Lemma 1. *Suppose that* $b_k - a_k = O\big(\sqrt{k}\big)$. *Then for all* $1\le i<j$ *we have*

$$|\mathbf{cov}(\alpha_i,\alpha_j)| \le C\left(\frac{i^{1/2}}{(j-i)^{1/2}} + \frac{1}{p_j (j-i)^{1/2}}\right)$$

Proof of Theorem 3. Similarly to the proof of Theorem 2.5 in [3], we first suppose $b_k - a_k \le C\sqrt{k}$. We are going to show that

$$\lim_{n\to\infty} \frac{1}{l(n)} \sum_{k=1}^{n} \frac{\xi_k}{k} = 0$$

with probability 1, where $\xi_k = \alpha_k - 1$. To this end, we proceed in the same way as in the proof of Theorem 2. We divide the sum into two, according to the size of summands. To the large summands we apply Theorem 1, while the other part proves to be so small that even a convergent series with finite expectation can be composed from it. Thus, let us define

$$T(n) = \frac{1}{l(n)} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}\left\{p_k\sqrt{k} < l_2^2(k)\right\} \xi_k ,$$

$$U(n) = \frac{1}{l(n)} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}\left\{p_k\sqrt{k} \geq l_2^2(k)\right\} \xi_k = \frac{1}{l(n)} \sum_{k=1}^{n} \frac{\xi_k'}{k} .$$

We will show that both sequences converge to 0 a.s.

We first deal with the convergence of $U(n)$. So as to apply Theorem 1, we have to find an appropriate function h. By Lemma 1, for $1 \leq i < j$ we can write

$$\begin{aligned} \mathbf{E}(\xi_i'\xi_j') &= \mathbf{I}\left\{p_i\sqrt{i} \geq l_2^2(i)\right\} \mathbf{I}\left\{p_j\sqrt{j} \geq l_2^2(j)\right\} \mathbf{cov}(\alpha_i, \alpha_j) \\ &\leq C\, \mathbf{I}\left\{p_i\sqrt{i} \geq l_2^2(i)\right\} \mathbf{I}\left\{p_j\sqrt{j} \geq l_2^2(j)\right\} \left(\frac{i^{1/2}}{(j-i)^{1/2}} + \frac{1}{p_j(j-i)^{1/2}}\right) \\ &\leq C \left(\frac{1}{(z-1)^{1/2}} + \frac{z^{1/2}}{l_2^2(z)(z-1)^{1/2}}\right) \\ &\leq C \left(\frac{1}{l_2^2(z)} + \frac{1}{(z-1)^{1/2}}\right) =: h(z) , \end{aligned} \tag{36}$$

where $z = j/i$. Note that the last line of (36) is already a decreasing function of z.

Next, we define $S(i)$. By definition of ξ_i' we have

$$\mathbf{E}(\xi_i'^2) = \mathbf{I}\left\{p_i\sqrt{i} \geq l_2^2(i)\right\} \mathbf{Var}(\alpha_i) = \mathbf{I}\left\{p_i\sqrt{i} \geq l_2^2(i)\right\} \left(\frac{1}{p_i} - 1\right) \leq \frac{i^{1/2}}{l_2^2(i)} =: S(i) .$$

It is not hard to see that these h and S satisfy the conditions of Theorem 1. Namely, (6), (7), and (9) are implied by the following estimations, respectively.

$$\int_1^\infty h(z)\frac{l_3(z)}{zl(z)}\,\mathrm{d}z \leq C \int_1^\infty \frac{l_3(z)}{zl(z)l_2^2(z)}\,\mathrm{d}z + C \int_1^\infty \frac{l_3(z)}{zl(z)(z-1)^{1/2}}\,\mathrm{d}z < \infty ,$$

$$\int_0^1 h(z+1)\left(\log\frac{1}{z}\right)^{2(1+\varepsilon)}\mathrm{d}z \leq C \int_0^{1/2} \frac{1}{z^{1/2}}\left(\log\frac{1}{z}\right)^{2(1+\varepsilon)}\mathrm{d}z < \infty ,$$

$$\sum_{i=1}^\infty \frac{1}{i^2}\, S(i) < \sum_{i=1}^\infty \frac{1}{i^{3/2}} < \infty .$$

By Theorem 1 we finally obtain that $U(n) \to 0$ with probability 1 under the assumption $b_k - a_k \leq C\sqrt{k}$.

Next, we prove that $T(n)$ converges to 0 a.s. We show that the expectation

$$\mathbf{E}\left(\sum_{n=1}^{\infty} \max\Big\{|T(k)| : \exp\left(2^n\right) \le k < \exp\left(2^{n+1}\right)\Big\}\right) \tag{37}$$

is finite. Now, (37) is clearly less than

$$\mathbf{E}\left(\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{e^{2^{n+1}}} \frac{1}{k} \mathbf{I}\left\{p_k\sqrt{k} < l_2^2(k)\right\} |\xi_k|\right).$$

Here $\mathbf{E}|\xi_k| \le 2$, hence, by interchanging the order of expectation we obtain that (37) is majorized by

$$C\sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}\left\{p_k\sqrt{k} < l_2^2(k)\right\} \sum_{2^{n+1}\ge \log k} \frac{1}{2^n} \le C\sum_{k=1}^{\infty} \frac{1}{k\log k} \mathbf{I}\left\{p_k\sqrt{k} < l_2^2(k)\right\}$$
$$\le C\sum_{p_k\ne 0} \frac{l_2^2(k)}{p_k\, k^{3/2}\log k},$$

which is finite by supposition. So we have

$$\lim_{n\to\infty} \frac{1}{l(n)} \sum_{k=1}^{n} \frac{\alpha_k - 1}{k} = 0$$

with probability 1, under the condition $b_k - a_k \le C\sqrt{k}$. From this point on the proof of Theorem 2.5 in [3] can be repeated to relax the restriction on $b_k - a_k$. □

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