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ON THE MULTIPLICITY OF THE SAMPLE MAXIMUM AND THE LONGEST HEAD RUN

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Dedicated to Endre Csáki on the occasion of his 65th birthday

ABSTRACT. Tossing a (not necessarily unbiased) coin n times let us denote the length of the longest head run by Z_n and the number of head runs of such length by M_n. Once Erdős asked about the asymptotic behavior of M_n as $n \to \infty$, and these questions motivated the problems that will be discussed in the present paper.

In an array of a double sequence of integer valued random variables, i.i.d. within rows, let $\mu(n)$ denote the multiplicity of the maximal value in the nth row. In Section 2 the asymptotic distribution of $\mu(n)$ is computed. Though limit distribution does not exist in the ordinary sense, a.s. limit distribution does, as proved in Section 3.

In Section 4 the multiplicity M_n of the maximal run is investigated in a general model of waiting times. By applying the results of Sections 2 and 3 an asymptotic formula is derived for the distribution of M_n, together with an a.s. limit distribution theorem.

Two interesting examples are discussed in Section 5. One of them is the motivating problem of longest head run, with a generalization of allowing at most d tails in between. The other one concerns the longest flat segment (or tube, in other words) of a (discrete) random walk.

The last section contains multivariate extensions.

1. INTRODUCTION

Let $X_1, X_2, \dots$ be an infinite sequence of i.i.d. Bernoulli random variables, with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p = q$, $0 < p < 1$. We can think of the X's as successive tosses by a (not necessarily unbiased) coin, interpreting 1's as heads and 0's as tails. Let us denote the length of the longest head run up to n, and the multiplicity of the longest head run by Z_n, and M_n, resp. That is,

$$Z_n = \max\left\{m - k : 0 \le k \le m \le n, X_{k+1} = \dots = X_m = 1\right\},$$

$$M_n = \sum_{i=0}^{n-Z_n} I\left(X_{i+1} = \dots = X_{i+Z_n} = 1\right).$$

The problem of characterizing the limit properties of M_n was posed in [6], Problem 2 on p. 62, see also Problem 11 of [3]. In the present paper we are going to

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show that M_n does not have a limit distribution in the ordinary sense, but it still has an a.s. limit distribution. Some results will also be presented on the a.s. limsup behaviour of the sequence M_n.

Since the lengths of disjoint head runs are independent geometrically distributed random variables with parameter q, it is quite natural to begin with the multiplicity of the maximum of a random sample from geometric distribution. Results of that type are found in [1], where the asymptotic form of the probability that the sample maximum is unique (multiplicity equals 1) is studied. Here we need a bit more: on the one hand we want to estimate the whole distribution of the maximum, and on the other hand, we also have to deal with joint distribution of maxima corresponding to different sample sizes. Results on the maximum of a sample from geometric distribution can easily be generalized to other discrete distributions with stabilizing hazard, and in the paper this general setting will be considered whenever it does not cause too much complication.

2. Multiplicity of sample maxima: asymptotic distribution

In this section we consider a general scheme comprising an array (double sequence) of random variables, i.i.d. within rows. That is, for every positive integer n let $Y_{1,n}, Y_{2,n}, \ldots, Y_{N,n}$ be a random sample of size $N = N_n$ from a nonnegative integer valued probability distribution $P(Y_{i,n} = k) = p_{k,n},\ k = 0, 1, \ldots$, and let $q_{k,n} = P(Y_{i,n} \geq k) = p_{k,n} + p_{k+1,n} + \cdots$. Suppose N tends to infinity increasingly with n. Consider the sequence of sample maxima

$$W_n = \max\{Y_{1,n}, \ldots, Y_{N,n}\},$$

and let $\mu(n)$ denote the multiplicity of W_n, that is, the number of sample elements $Y_{i,n},\ i \leq N$, being equal to W_n.

Lemma 2.1.

$$P(\mu(n) = m) = \frac{1}{m!} \sum_{k=0}^{\infty} (N_n p_{k,n})^m \exp(-N_n q_{k,n}) + o(1), \tag{2.1}$$

as $n \to \infty$.

Proof. For $1 \leq m \leq N$ clearly

$$\begin{aligned} P(\mu(n) = m) &= \sum_{k=0}^{\infty} P(\mu(n) = m, W_n = k) \\ &= \sum_{k=0}^{\infty} \binom{N}{m} p_{k,n}^m (1 - q_{k,n})^{N-m}. \end{aligned} \tag{2.2}$$

Let $\frac{1}{2} < c < 1$, and $k(n) = \max\{k : q_{k,n} \geq N^{-c}\}$. Then all terms on the right hand sides of (2.1) and (2.2) with $k \leq k(n)$ are asymptotically negligible, since

$$\sum_{k \leq k(n)} (Np_{k,n})^m \exp(-Nq_{k,n}) \leq N^m \exp\left(-N^{1-c}\right) \sum_{k \leq k(n)} p_{k,n}^m \leq N^m \exp\left(-N^{1-c}\right),$$

and similarly,

$$\sum_{k\le k(n)} \binom{N}{m} p_{k,n}^m \left(1-q_{k,n}\right)^{N-m} \le \frac{N^m}{m!} \exp\left(-(N-m)N^{-c}\right) \sum_{k\le k(n)} p_{k,n}^m = o(1).$$

On the other hand, corresponding terms of the two series above are asymptotically equal as $n\to\infty$, besides, that holds uniformly for $k>k(n)$.

$$\begin{aligned}\binom{N}{m} p_{k,n}^m \left(1-q_{k,n}\right)^{N-m} &\le \left(1-q_{k(n)+1,n}\right)^{-m} \frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right)\\ &\le \left(1-N^{-c}\right)^{-m} \frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right),\\ &= \left(1+o(1)\right) \frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right),\end{aligned}$$

and

$$\begin{aligned}\binom{N}{m} p_{k,n}^m \left(1-q_{k,n}\right)^{N-m} \ge \left(1-\frac{m}{N}\right)^m &\left[\left(1-q_{k(n)+1,n}\right)\exp\left(q_{k(n)+1,n}\right)\right]^N \times\\ &\times \frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right)\\ \ge \left(1-\frac{m}{N}\right)^m &\left(1-\frac{1}{N^{2c}}\right)^N \frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right)\\ = \left(1-o(1)\right) &\frac{1}{m!} \left(Np_{k,n}\right)^m \exp\left(-Nq_{k,n}\right).\end{aligned}$$

In the last inequality we used the fact that $(1-x)e^x \ge 1-x^2$.

Remark 2.1. From the proof it is clear that the order of magnitude of the remainders in (2.1) is $O\left(N_n^{-1}\right)$, and this cannot be improved for $m>1$.

Let us introduce the (discrete) hazard function

$$r_{k,n} = p_{k,n}/q_{k,n} = P\left(Y_{i,n}=k \mid Y_{i,n}\ge k\right).$$

Clearly,

$$q_{k,n} = \prod_{i=0}^{k-1}\left(1-r_{i,n}\right), \quad p_{k,n} = r_{k,n}\prod_{i=0}^{k-1}\left(1-r_{i,n}\right).$$

When all variables $Y_{i,n}$ are identically distributed, we can suppress n in the subscripts of $p_{k,n}$, $q_{k,n}$, and $r_{k,n}$. In the case of geometric distribution defined by $p_k = qp^{k-1}$, $k\ge 1$, where $0<q<1$ is a parameter and $p+q=1$, we have $q_k = p^{k-1}$ and the hazard is constant: $r_k = q$. We say that a distribution $\{p_k,\, k\ge 0\}$ is of *stabilizing hazard*, if $r_k\to r$ as $k\to\infty$, where $0<r<1$. Equivalently, we can also write $p_{k+1}/p_k \to r$. Well-known distributions of this type are, for instance, the negative binomial distributions of arbitrary order, and the logarithmic distribution.

For the sequel we need to define a similar property for arrays. Requiring that all of the row distributions are of stabilizing hazard, uniformly in n, and with the same limit r, is certainly sufficient, but a somewhat weaker condition will also do, if we add that the sequence of row distributions should increase stochastically.

Definition 2.1. Assume that $q_{k,n}$ is an increasing function of n for every $k = 1, 2, \dots$. Let $\kappa(n) = \min\{k : N_n q_{k,n} \le 1\}$, then $\kappa(n) \to \infty$ increasingly. We say that the array $\{Y_{i,n} : 1 \le i \le N_n, n \ge 1\}$ possesses *the property of stabilizing hazard* (SH, briefly), if

$$\lim_{n\to\infty} r_{\kappa(n)+k,n} = r \in (0,\, 1)$$

for arbitrary integer k.

This property enables us to derive a simple asymptotic formula for the distribution of $\mu(n)$. It will turn out that limit distribution does not exist in the ordinary sense, but in Section 3 we will show that $\mu(n)$ does possess a.s. limit distribution, namely a logarithmic one.

Let us define the functions f_m, $m = 1, 2, \dots$ by the doubly infinite series

$$f_m(x,y) = \frac{(1-y)^m}{m!} \sum_{k=-\infty}^{+\infty} \left(y^k x\right)^m \exp\left(-y^k x\right), \quad 0 < x,\ 0 < y < 1. \tag{2.3}$$

It is easy to see that $f_m(xy, y) = f_m(x, y)$. Hence the series (2.3) is uniformly convergent in the stripe $0 < x$, $a \le y \le b$ $(0 < a < b < 1)$, because

$$\max_{a\le y\le b,\, y\le x\le 1} \left(y^k x\right)^m \exp\left(-y^k x\right) \le \begin{cases} b^{km} \exp\left(-a^{k+1}\right), & \text{if } k \ge 0, \\ a^{km} \exp\left(-b^{k+1}\right), & \text{if } k < 0, \end{cases}$$

which makes a convergent series. Thus $f_m(x, y)$ is continuous.

Now we show that $\sum_{m=1}^{\infty} f_m(x, y) = 1$. By changing the order of summation we have

$$\begin{aligned} \sum_{m=1}^{\infty} f_m(x,y) &= \sum_{m=1}^{\infty} \frac{(1-y)^m}{m!} \sum_{k=-\infty}^{+\infty} \left(y^k x\right)^m \exp\left(-y^k x\right) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{m=1}^{\infty} \frac{1}{m!} \left[(1-y) y^k x\right]^m \exp\left(-y^k x\right) \\ &= \sum_{k=-\infty}^{+\infty} \exp\left(-y^k x\right) \left[\exp\left((1-y) y^k x\right) - 1\right] \\ &= \sum_{k=-\infty}^{+\infty} \left[\exp\left(-y^{k+1} x\right) - \exp\left(-y^k x\right)\right] \\ &= \lim_{k\to+\infty} \exp\left(-y^k x\right) - \lim_{k\to-\infty} \exp\left(-y^k x\right) \\ &= 1. \end{aligned}$$

Lemma 2.2. *Suppose SH holds. Let* $c(n) = N_n q_{\kappa(n),n}$, *then*

$$P\left(\mu(n) = m\right) = f_m\left(c(n), 1 - r\right) + o(1), \tag{2.4}$$

as $n \to \infty$.

Proof. Let M be a fixed positive integer. From (2.1) it follows for all sufficiently large n that

$$P\left(\mu(n) = m\right) \ge \frac{1}{m!} \sum_{k=-M}^{M} \left(N_n p_{\kappa(n)+k,n}\right)^m \exp\left(-N_n q_{\kappa(n)+k,n}\right) + o(1).$$

Since $N_n p_{\kappa(n)+k,n} \sim r(1-r)^k c(n)$ and $N_n q_{\kappa(n)+k,n} - (1-r)^k c(n) \to 0$ as $n \to \infty$, we obtain that

$$\liminf_{n\to\infty} \Big[P\left(\mu(n) = m\right) - \frac{1}{m!} \sum_{k=-M}^{M} \left[r(1-r)^k c(n)\right]^m \exp\left(-(1-r)^k c(n)\right) \Big] \geq 0$$

for every $M > 0$, hence

$$\liminf_{n\to\infty} \Big[P\left(\mu(n) = m\right) - f_m\left(c(n),\, 1-r\right) \Big] \geq 0, \quad m = 1,\, 2,\, \ldots.$$

On the other hand,

$$\begin{aligned}
&\limsup_{n\to\infty} \Big[P\left(\mu(n) = m\right) - f_m\left(c(n),\, 1-r\right) \Big] \\
&= \limsup_{n\to\infty} \sum_{1\leq j\neq m} \Big[f_j\left(c(n),\, 1-r\right) - P\left(\mu(n) = j\right) \Big] \\
&\leq \sum_{1\leq j\neq m} \limsup_{n\to\infty} \Big[f_j\left(c(n),\, 1-r\right) - P\left(\mu(n) = j\right) \Big] \\
&= - \sum_{1\leq j\neq m} \liminf_{n\to\infty} \Big[P\left(\mu(n) = j\right) - f_j\left(c(n),\, 1-r\right) \Big] \leq 0,
\end{aligned}$$

completing the proof.

Lemma 2.3. *Suppose SH holds, and, in addition,*

$$\lim_{n\to\infty} \frac{N_n}{N_{n+1}} = 1, \quad \textit{and} \quad \lim_{n\to\infty} \frac{q_{\kappa(n),n}}{q_{\kappa(n),n+1}} = 1 \tag{2.5}$$

(this is the case, for instance, if the n-th row of the array is built up from the first n terms of the same i.i.d. sequence). Then the set of limit points of the sequence $\{c(n),\, n \geq 1\}$ coincides with the closed interval $[1-r,\, 1]$.

Proof. Let $\nu(k) = \max\{n : N_n q_{k,n} \leq 1\}$, that is,

$$N_{\nu(k)} q_{k,\nu(k)} \leq 1 < N_{\nu(k)+1} q_{k,\nu(k)+1}.$$

Then $\nu(k)$ is increasing; and for n satisfying $\nu(k-1) < n \leq \nu(k)$ we have $\kappa(n) = k$, because

$$N_n q_{k,n} \leq N_{\nu(k)} q_{k,\nu(k)} \leq 1 < N_{\nu(k-1)+1} q_{k-1,\nu(k-1)+1} \leq N_n q_{k-1,n}.$$

Consequently, $c(n) \leq c(n+1)$ for $\nu(k-1) < n < \nu(k)$.

Now we show that $c(n)$ crawls across the interval $[1-r,\, 1]$, as n runs from $\nu(k-1)+1$ to $\nu(k)$. First, let $n = \nu(k-1)+1$. Then $\kappa(n) - 1 = k - 1 = \kappa(n-1)$, hence

$$\begin{aligned}
c(n) &= N_n q_{k,n} = N_n q_{\kappa(n)-1,n} \left(1 - r_{\kappa(n)-1,n}\right) > 1 - r_{\kappa(n)-1,n} \sim 1 - r, \\
c(n) &= \left(1 - r_{\kappa(n)-1,n}\right) \frac{N_n}{N_{n-1}} \frac{q_{\kappa(n)-1,n}}{q_{\kappa(n)-1,n-1}}\, c(n-1) \\
&\leq \left(1 - r_{\kappa(n)-1,n}\right) \frac{N_n}{N_{n-1}} \frac{q_{\kappa(n-1),n}}{q_{\kappa(n-1),n-1}} \sim 1 - r,
\end{aligned}$$

as $k \to \infty$.

Secondly, let $\nu(k-1) < n < \nu(k)$. Then, by (2.5),

$$\frac{c(n)}{c(n+1)} = \frac{N_n q_{k,n}}{N_{n+1} q_{k,n+1}} = \frac{N_n}{N_{n+1}} \cdot \frac{q_{\kappa(n),n}}{q_{\kappa(n),n+1}} \to 1$$

as $k \to \infty$.

Thirdly, let $n = \nu(k)$. Then

$$c(n) = \frac{N_n}{N_{n+1}} \frac{q_{\kappa(n),n}}{q_{\kappa(n),n+1}} N_{n+1} q_{\kappa(n+1)-1,n+1} > \frac{N_n}{N_{n+1}} \frac{q_{\kappa(n),n}}{q_{\kappa(n),n+1}} \to 1,$$

as $k \to \infty$; and at the same time $c(n) \le 1$ by definition.

From all these one can readily conclude that $c(n)$ oscillates between $1-r$ and 1.

As a corollary we obtain that $\mu(n)$ does not converge in distribution.

3. Multiplicity of sample maxima: a.s. limit distribution

From Lemma 2.2 and Lemma 2.3 it is clear that in the case of stabilizing hazard $\mu(n)$ is stochastically bounded, but very often it does not have a limit distribution as $n \to \infty$, because of the logarithmic periodicity appearing in the asymptotic distribution.

Such a periodicity can be eliminated by a sufficiently strong (e.g. logarithmic) summation procedure, and even the existence of an a.s. limit distribution can often be proved.

A sequence of random variables ζ_n is said to have a.s. limit distribution, if for every real x

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} I(\zeta_n \le x) = G(x) \quad \text{a.s.} \tag{3.1}$$

with some non-degenerate distribution function $G(x)$. Under quite general conditions, (3.1) holds if and only if the sequence of probabilities $P(\zeta_n \le x)$ is logarithmically summable to $G(x)$, as it is implied by the following simple lemma.

Lemma 3.1. [4] *Let $\xi_1, \xi_2, \dots$ be a sequence of uniformly bounded random variables (e.g. $\xi_n = I(\zeta_n \le x) - P(\zeta_n \le x)$), such that $|E(\xi_i \xi_j)| \le h(j/i)$, $1 \le i < j$, where h is a positive decreasing function, and*

$$\int_1^\infty \frac{h(x)}{x \log x}\, dx \le \infty.$$

Then

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} \xi_n = 0 \quad \textit{a.s.}$$

Arrays are not really suitable for investigations of a.s. type, because the joint distributions of the rows are indetermined. Therefore, in this section we will confine ourselves to a single series of i.i.d. random variables $Y_1, Y_2, \dots$; thus $\mu(n)$ is the multiplicity of the maximal value among the first n variables. In other words, here we consider a special array with $N_n = n$ and $Y_{i,n} = Y_i$. According to this, the row index n in the subscripts of $p_{k,n}$, $q_{k,n}$, and $r_{k,n}$ will be suppressed. We will require that the distribution of Y_i is of stabilizing hazard. Consequently, our special array satisfies the SH property.

In order to apply Lemma 3.1 we first have to estimate $P\left(\mu(n) = m,\, \mu(s) = m\right)$, $n < s$. Let $\mu(n,s) = \#\left\{i : n < i \le s,\ Y_i = \max\{Y_{n+1}, \ldots, Y_s\}\right\}$.

Lemma 3.2. *Suppose* $r_k \to r \in (0,\,1)$. *For arbitrary Borel sets* A, B, *and positive integers* $n < s$ *we have*

$$\left|P\left(\mu(n) \in A,\, \mu(s) \in B\right) - P\left(\mu(n) \in A\right) P\left(\mu(s) \in B\right)\right| \le C\,\frac{n}{s},$$

where the constant C *only depends on the distribution of* Y_i.

Proof. Clearly,

$$\begin{aligned}
&\left|P\left(\mu(n) \in A,\, \mu(s) \in B\right) - P\left(\mu(n) \in A\right) P\left(\mu(s) \in B\right)\right| \le \\
&\le \left|P\left(\mu(n) \in A,\, \mu(s) \in B\right) - P\left(\mu(n) \in A,\, \mu(n,s) \in B\right)\right| + \\
&\qquad + \left|P\left(\mu(n) \in A\right) P\left(\mu(n,s) \in B\right) - P\left(\mu(n) \in A\right) P\left(\mu(s) \in B\right)\right| \\
&\le 2P\left(\mu(n,s) \ne \mu(s)\right).
\end{aligned}$$

If $\mu(n,s) \ne \mu(s)$, there must be a sample element of maximum value among the first n ones. By symmetry, the maximal elements are distributed uniformly among the sample, thus we can write

$$\begin{aligned}
P\left(\mu(n,s) \ne \mu(s)\right) &\le P\left(\exists i \le n : Y_i = W_s\right) \le nP\left(Y_1 = W_s\right) = \\
&= n\sum_{k=0}^{\infty} p_k \left(1 - q_{k+1}\right)^{s-1} = \frac{n}{s} \sum_{k=0}^{\infty} s p_{k+1} \left(1 - q_{k+1}\right)^{s-1} \frac{r_k}{r_{k+1}(1 - r_k)}.
\end{aligned} \tag{3.2}$$

The sum in (3.2) is very similar to that we obtained for $P\left(\mu(s) = 1\right)$ in (2.2), the only difference is that each term is multiplied by a factor. In the case of stabilizing hazard those factors converge to $(1-r)^{-1}$, thus they are bounded. Hence

$$P\left(\mu(n,s) \ne \mu(s)\right) \le C\,\frac{n}{s}. \tag{3.3}$$

Remark 3.1. In the case of stabilizing hazard it sounds plausible that $E\mu(s)$ remains bounded as $s \to \infty$, but let us notice that from the proof above it follows that

$$\limsup_{s\to\infty} E\mu(s) \le (1-r)^{-1}.$$

Indeed, since $\mu(s)$ can be decomposed into a sum of interchangeable indicators, we have $E\mu(s) = sP\left(Y_1 = W_s\right)$, that is, $E\mu(s)$ is just equal to the series in (3.2). In the proof of Lemma 2.1 we have pointed out that the beginning of the series (2.2) becomes negligible as $s \to \infty$, hence it follows that the lim sup of the sum in (3.2) does not exceed $(1-r)^{-1}$.

Now we are in a position to prove the main result of this section.

Theorem 3.1. *With probability* 1

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} I\left(\mu(n) = m\right) = \frac{r^m}{m \log\left(\frac{1}{1-r}\right)}, \quad m = 1, 2, \ldots .$$

Proof. We can apply Lemma 3.1 with $h(x) = Cx^{-1}$. By Lemma 2.2 we have

$$\frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} P\left(\mu(n) = m\right) = \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} f_m(nq_{\kappa(n)},\, 1-r) + o(1).$$

According to the notation used in the proof of Lemma 2.3, let $\nu(k)$ denote the integer part of $1/q_k$. Then $\nu(k)/\nu(k+1) \to 1-r$, hence $\log \nu(k) \sim k \log \frac{1}{1-r}$, as $k \to \infty$; and for $k = 1, 2, \ldots$

$$S(k) =: \sum_{n=\nu(k-1)+1}^{\nu(k)} \frac{1}{n} f_m(nq_{\kappa(n)}, 1-r) = \sum_{n=\nu(k-1)+1}^{\nu(k)} \frac{1}{n} f_m(nq_k, 1-r).$$

Thus $S(k)$ appears to be an integral approximating sum. Remembering the properties of the function $f_m(x,y)$ we get

$$\begin{aligned} S(k) &= \int_{\nu(k-1)q_k}^{\nu(k)q_k} \frac{1}{x} f_m(x, 1-r)dx + o(1) \\ &= \int_{1-r}^{1} \frac{1}{x} f_m(x, 1-r)dx + o(1) := \alpha + o(1) \end{aligned}$$

as $k \to \infty$. Since for $\nu(j-1) \le t < \nu(j)$ we can write

$$\frac{1}{\log \nu(j)} \sum_{k=1}^{j-1} S(k) \le \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} f_m(nq_{\kappa(n)}, 1-r) \le \frac{1}{\log \nu(j-1)} \sum_{k=1}^{j} S(k),$$

it follows that

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n} P\left(\mu(n) = m\right) = \frac{\alpha}{\log \frac{1}{1-r}}.$$

Let us compute α.

$$\begin{aligned} \alpha &= \int_{1-r}^{1} \frac{r^m}{m!} \sum_{k=-\infty}^{+\infty} (1-r)^{km} x^{m-1} \exp\left(-(1-r)^k x\right) dx \\ &= \frac{r^m}{m} \sum_{k=-\infty}^{+\infty} \int_{1-r}^{1} \frac{1}{(m-1)!} (1-r)^{km} x^{m-1} \exp\left(-(1-r)^k x\right) dx \end{aligned}$$

$$= \frac{r^m}{m} \sum_{k=-\infty}^{+\infty} \int_{(1-r)^{k+1}}^{(1-r)^k} \frac{1}{(m-1)!}\, y^{m-1} e^{-y} dy$$

$$= \frac{r^m}{m} \int_0^\infty \frac{1}{(m-1)!}\, y^{m-1} e^{-y} dy$$

$$= \frac{r^m}{m}.$$

By virtue of Lemma 3.1 the proof is completed.

4. Multiplicity of maximal runs

It is quite easy to see that the (random) number of head runs up to n is asymptotically npq, thus the multiplicity of the longest head run is approximately the same as that of the maximum of a sample of size npq, drawn from geometric distribution with parameter q. Similar, but somehow refined, approach can be applied in the following more general setting.

Let $(\mathcal{X}, \mathcal{F})$ be a measurable space and $X_1, X_2, \ldots$ i.i.d. $\mathcal{X}$-valued random variables with distribution Q. Let $X_{n,k}$ denote the block $(X_n, X_{n+1}, \ldots, X_{n+k-1})$. Suppose for every positive integer k we are given a measurable set $B_k \subset \mathcal{F}^k$ such that

$$B_k \subset B_{k-1} \times \mathcal{X}, \qquad B_k \subset \mathcal{X} \times B_{k-1}.$$

Let $A_{i,j}$ abbreviate the event $\{X_{i,j} \in B_j\}$; if $A_{i,j}$ occurs, we say that *a run of length* j begins at i. Define

$$T_k = \min\{n \ge k : A_{n-k+1,k} \text{ occurs}\};$$

then $T_1 < T_2 < \cdots$. Let $p(k) = P(A_{1,k}) = P(T_k = k)$; this is decreasing in k. Assume that $p(k) > 0$ for every k; then T_k is finite and it has finite moments of arbitrary order. Particularly, denote ET_k shortly by $E(k)$.

In [5] it is shown that

$$\left(1 - \frac{1}{E(k)}\right)^n [1 - kp(k)] - 2kE(k)p(k)^2 \le P(T_k \ge n+k) \le \left(1 - \frac{1}{E(k)}\right)^n \tag{4.1}$$

(see Lemma 2.2 there). These inequalities will prove to be very useful in the sequel.

Finally, let us introduce $Z_n = \max\{k : T_k \le n\}$ (this corresponds to the length of the longest head run), and its multiplicity $M_n = \#\{i \le n - Z_n + 1 : X_{i,Z_n} \in B_{Z_n}\}$. More generally, for $n < s$ let

$$Z_{n,s} = \max\{k \le s - n : X_{i,k} \in B_k \text{ for some } n < i \le s - k + 1\},$$

and let $M_{n,s}$ be its multiplicity.

In [5] it was proved under quite general conditions that Z_n has an a.s. limit distribution, though its asymptotic distribution shows logarithmic periodicity. Here we are going to prove the same for M_n. To do so we will require the following conditions.

$$\lim_{k\to\infty} p(k+1)/p(k) = p \in (0,\, 1), \tag{4.2}$$

$$\lim_{k\to\infty} P\left(A_{1,k+1} \mid A_{1,k} \cap A_{j+1,k-j+1}\right) = 1, \quad j = 1, 2, \dots\ . \tag{4.3}$$

In [5] (case (ii) of Theorem 3.1) a condition, very similar to (4.2), was introduced, namely, the existence of a positive, increasing, differentiable function f such that $E(k) \sim f(k)$, the limit $c := \lim_{t\to\infty} (\log f(t))'$ exists, and it is positive and finite. Essentially, this means that $E(k)/E(k+1) \to e^{-c}$. As we can see from Lemma 4.1 below, this latter is implied by our conditions (4.2) and (4.3), too. The meaning of condition (4.3) is that the occurrences of the maximal run are all disjoint with probability tending to 1 as $n \to \infty$, see the proof of Lemma 4.2.

In applications it is often much easier to compute $p(k)$ than $E(k)$ itself. Though it is not so hard to see that $1 \le p(k)E(k) \le k$ (see Lemma 2.1 of [5]), (4.2) and (4.3) provide more precise relation between $p(k)$ and $E(k)$.

Lemma 4.1.

$$\lim_{k\to\infty} p(k)E(k) = \frac{1}{1-p},$$

and hence $E(k)/E(k+1) \to p$.

Proof. Let us introduce $\delta = \delta(\ell, j)$ as

$$\delta = \sup\left\{P\left(\overline{A}_{1,k+1} \mid A_{1,k} \cap A_{t+1,k-t+1}\right) : 1 \le t \le j,\ \ell \ge k\right\}.$$

Clearly, $\lim_{k\to\infty} \delta(k, j) = 0$ by (4.3) for every $j = 1, 2, \dots$; in addition, we can write

$$p(k+1) \le P\left(A_{1,k} \cap A_{t+1,k-t+1}\right) \le \frac{p(k+1)}{1-\delta(k,j)}, \quad 1 \le t \le j. \tag{4.4}$$

Let us apply the following corollary of Lemma 2.1 of [5]:

$$\lim_{k\to\infty} p(k)E(k) = \lim_{j\to\infty}\lim_{k\to\infty} \frac{p(k)}{P\left(T_k = k+j\right)}. \tag{4.5}$$

Here we have

$$\begin{aligned} P\left(T_k = k+j\right) &= P\left(\overline{A}_{1,k} \cap \dots \cap \overline{A}_{j,k} \cap A_{j+1,k}\right) \\ &\le P\left(A_{j+1,k}\right) - P\left(A_{j+1,k} \cap A_{j,k+1}\right) \le p(k) - p(k+1). \end{aligned}$$

On the other hand,

$$\begin{aligned} P(T_k = k+j) &\ge P\left(A_{j+1,k}\right) - P\Big(\bigcup_{t=1}^{j-1} \left(A_{j+1,k} \cap A_{t,k} \cap \overline{A}_{t+1,k}\right) \bigcup \left(A_{j+1,k} \cap A_{j,k}\right)\Big) \\ &\ge P\left(A_{1,k}\right) - \sum_{t=1}^{j-1} P\left(A_{j-t+2,k-j+t} \cap A_{1,k} \cap \overline{A}_{1,k+1}\right) - P\left(A_{1,k} \cap A_{2,k}\right) \\ &\ge p(k+1) - (j-1)\frac{\delta}{1-\delta}\, p(k+1) - \frac{1}{1-\delta}\, p(k+1) \\ &= p(k) - p(k+1) - \frac{j\delta}{1-\delta}\, p(k+1). \end{aligned}$$

These inequalities, together with (4.2) and (4.5), give the result to be proved.

In order to compute the asymptotic distribution of M_n let us define an array to which the results of Section 2 can be applied. Let N_n be an increasing sequence of positive integers, such that n/N_n is increasing, and

$$\lim_{n\to\infty} N_n = +\infty, \qquad \lim_{n\to\infty} N_n/N_{n+1} = 1, \qquad N_n = o\left(n/\log n\right). \tag{4.6}$$

Let R_n denote the integer part of n/N_n; and let us divide the sequence $X_1, \ldots, X_n$ into N_n blocks of size R_n, plus a last one of length $< N_n$, if N_n does not divide n. Define $Y_{i,n}$ as the length of the maximal run observed in the ith block, that is, $Y_{i,n} = Z_{(i-1)R_n, iR_n}$, $i = 1, 2, \ldots, N_n$. For n fixed, these are i.i.d. random variables.

Lemma 4.2. *With the notation of Section 2 we have*

$$\lim_{n\to\infty} P\left(W_n = Z_n,\, \mu(n) = M_n\right) = 1.$$

Proof. Let us see how it can happen that $W_n \neq Z_n$ or $\mu(n) \neq M_n$. In that case at least one occurrence of the maximal run is not counted, because either (a) it crosses the boundary between two adjacent blocks, (b) it reaches the last, not counted block, (c) there is a block with at least two disjoint occurrences of the maximal run, or (d) there are two overlapping occurrences somewhere.

Suppose the length of the maximal run is m. Then the probability of case (a) is estimated by $(N_n - 1)\,(m-1)p(m)$; that of case (b) by $(N_n - 1)p(m)$; and the probability of case (c) is less than $N_n R_n^2 p(m)^2$. Finally, in case (d) the probability of two overlapping occurrences is estimated by

$$\begin{aligned}
&\sum_{t=1}^{m-1} nP\left(A_{1,m} \cap A_{t+1,m} \cap \overline{A}_{1,m+1}\right) \\
&\leq n \sum_{t=1}^{j} \frac{p(m+1)\delta(m,j)}{1-\delta(m,j)} + n \sum_{t>j} p(t)p(m) \\
&= np(m+1)\frac{j\delta(m,j)}{1-\delta(m,j)} + np(m)\sum_{t>j} p(t),
\end{aligned}$$

for arbitrary j. Here we applied (4.4) to terms with $t \leq j$.

Let $\varepsilon > 0$ be fixed, and choose j in such a way that $\sum_{t>j} p(t) < \varepsilon$. Further, let $K = K_n$ be the largest integer such that

$$\exp\left(-n/E(K)\right) \leq \varepsilon.$$

Then $K < C\log n$, and $np(K) \leq C \log \frac{1}{\varepsilon}$ by Lemma 4.1. (Here and in what follows the same letter C will denote different constants, all independent of ε.) In addition, by (4.1) we have

$$P\left(Z_n < K\right) = P\left(T_K > n\right) \leq \exp\left(-\frac{n-K}{E(K)}\right) \leq C\varepsilon.$$

Thus we can write

$$
\begin{aligned}
1-P\left(W_n=Z_n,\,\mu(n)=M_n\right) &\le P\left(Z_n<K\right)+\\
&+\sum_{m\ge K}\left[Nmp(m)+\frac{[np(m)]^2}{N}+np(m)\frac{j\delta(K,j)}{1-\delta(K,j)}+np(m)\sum_{t>j}p(t)\right]\\
&\le C\left[\varepsilon+NKp(K)+\frac{[np(K)]^2}{N}+np(K)j\delta(K,j)+np(K)\varepsilon\right]\\
&\le C\left[\varepsilon+\frac{N\log n}{n}\log\frac{1}{\varepsilon}+\frac{1}{N}\left(\log\frac{1}{\varepsilon}\right)^2+j\delta(K,j)\log\frac{1}{\varepsilon}+\varepsilon\log\frac{1}{\varepsilon}\right].
\end{aligned}
$$

Hence

$$\liminf_{n\to\infty} P\left(W_n=Z_n,\,\mu(n)=M_n\right)\ge 1-C\varepsilon\log\frac{1}{\varepsilon},$$

where ε can be arbitrarily small positive number.

Lemma 4.3. *The array $\{Y_{i,n}\}$ satisfies property SH with $r=1-p$.*

Proof. Since R_n is increasing, so is $Y_{1,n}$, hence

$$q_{k,n}=P\left(Y_{1,n}\ge k\right)\le P\left(Y_{1,n+1}\ge k\right)=q_{k,n+1}.$$

From (4.1) it is clear that $P\left(T_k\le n\right)\sim n/E(k)$, if k and n tend to infinity in such a way that $n/E(k)\to 0$ and $k=o(n)$. Therefore

$$q_{k,n}=P\left(Y_{1,n}\ge k\right)=P\left(T_k\le R_n\right)\sim\frac{n}{N_nE(k)} \tag{4.7}$$

if the right-hand side converges to 0, and $kN_n=o(n)$. From Definiton 2.1 one can immediately see that $\kappa(n)=O(\log n)$, hence $q_{\kappa(n),n}\sim\dfrac{n}{N_nE(\kappa(n))}$ by (4.6) and (4.7), and also $q_{\kappa(n)+k,n}\sim\dfrac{n}{N_nE(\kappa(n)+k)}$. Consequently, $c(n)\sim\dfrac{n}{E(\kappa(n))}$, and

$$1-r_{\kappa(n)+k,n}=\frac{q_{\kappa(n)+k+1,n}}{q_{\kappa(n)+k,n}}\sim\frac{E(\kappa(n)+k)}{E(\kappa(n)+k+1)}\sim p,$$

by (4.2) and Lemma 4.1.

By combining Lemmas 4.2, 4.3, 2.2 and 2.3 we can approximate the distribution of M_n.

Theorem 4.1. *Let $\lambda(n)=\min\{k:E(k)\ge n\}$. Then*

$$P\left(M_n=m\right)=f_m\Big(\frac{n}{E(\lambda(n))},\,p\Big)+o(1).$$

The set of limit points of $n/E(\lambda(n))$ is the whole interval $[p,1]$, thus $P\left(M_n=m\right)$ does not converge.

Proof. Let $d=\lambda(n)-\kappa(n)$; clearly, $|d|\le 1$ for all sufficiently large n. By (4.2), $n/E(\kappa(n)+d)\sim c(n)p^d$, but this change does not make any difference by the periodicity and continuity of f_m.

The second part of the Theorem will follow from Lemma 2.3 if we show that $q_{\kappa(n),n}\sim q_{\kappa(n),n+1}$, but this latter is implied by (4.7).

Let us pass over to the a.s. limit distribution. It does not appear easy to apply Theorem 3.1 directly; we are going to use Lemma 3.1 instead. Thus, we have to estimate $P\left(M_n=m,\,M_s=m\right)$, $n<s$. Define $\ell(x)=\max\{\log x,1\}$, $x>0$.

Lemma 4.4. *For arbitrary Borel sets A, B, and positive integers $n < s$ we have*

$$\left|P\left(M_n \in A,\, M_s \in B\right) - P\left(M_n \in A\right) P\left(M_s \in B\right)\right| \le C\, \frac{n}{s}\, \ell\Big(\frac{s}{n}\Big)\,.$$

Proof. Exactly in the same way as in the proof of Lemma 3.2 we obtain that

$$\left|P\left(M_n \in A,\, M_s \in B\right) - P\left(M_n \in A\right) P\left(M_s \in B\right)\right| \le 2P\left(M_{n,s} \ne M_s\right).$$

If $M_{n,s} \ne M_s$, there must be a maximal run beginning in course of the first n experiments. Suppose $Z_s = k$, then $T_k < n + k$ but $T_{k+1} > s$. Thus we can write

$$\begin{aligned} P\left(M_{n,s} \ne M_s\right) &\le P\Bigg(\bigcap_{k\ge 1} \left\{T_k < n+k,\, T_{k+1} > s\right\}\Bigg) \\ &\le P\left(T_{k_0} < n + k_0\right) + P\left(T_{k_0} > s\right), \end{aligned} \tag{4.8}$$

for arbitrary positive integer k_0. Let k_0 be the largest integer k such that

$$\frac{1}{E(k)} \ge \frac{1}{s}\, \ell\Big(\frac{s}{n}\Big)\,,$$

then $k_0 = O\left(\log s\right)$ by (4.2). Terms on the right-hand side of (4.8) are easy to estimate. On the one hand, by Lemma 4.1 we have

$$P\left(T_{k_0} < n + k_0\right) \le np(k_0) = O\left(\frac{n}{E(k_0)}\right) = O\left(\frac{n}{s}\, \ell\Big(\frac{s}{n}\Big)\right).$$

On the other hand, by applying (4.1) we can write

$$P\left(T_{k_0} > s\right) \le \exp\left(-\frac{s - k_0}{E(k_0)}\right) = O\left(\frac{n}{s}\right).$$

This completes the proof.

As a corollary, we immediately obtain the a.s. limit distribution of M_n.

Theorem 4.2. *With probability* 1

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n=1}^{t} \frac{1}{n}\, I\left(M_n = m\right) = \frac{(1-p)^m}{m \log(1/p)}\,, \quad m = 1, 2, \ldots .$$

A proof of this assertion can be given simply by copying that of Theorem 3.1, therefore it will be omitted.

5. Examples

In this section we specialize our results to obtain interesting corollaries in two important particular cases. Detailed discussion of these models are presented in [2], and also in [5], where the a.s. limit distribution of Z_n is derived.

5.1. The longest (d-interrupted) head run.

Let $X_1, X_2, \ldots$ be the Bernoulli sequence of the Introduction. For a fixed non-negative integer d define

$$B_k = \left\{(x_1, \ldots, x_k) \in \{0,\, 1\}^k : x_1 + \cdots + x_k \geq k - d\right\}.$$

Then Z_n is the length of the longest block up to n containig at most d zeros (tails), and M_n is its multiplicity. Clearly,

$$p(k) \sim \frac{1}{d!} \left(\frac{qk}{p}\right)^d p^k,$$

thus (4.2) holds. On the other hand, $P\left(A_{1,k} \cap A_{j+1,k-j+1} \cap \overline{A}_{1,k+1}\right) = 0$, if $d = 0$ (the case of pure head runs), and for $d \geq 1$ it is majorized by the probability that $X_{j+1,k-j}$ contains exactly $d-1$ zeros, which is approximately

$$\frac{1}{(d-1)!} \left(\frac{qk}{p}\right)^{d-1} p^{k-j} = o\left(p(k+1)\right)$$

as $k \to \infty$. Thus (4.3) is also satisfied, and Theorems 4.1 and 4.2 are valid.

Particularly, when $d = 0$, we have $n/E(\lambda(n)) \sim nqp^{\lambda(n)}$, therefore Theorem 4.1 simplifies to

$$P\left(M_n = m\right) = f_m\left(nq,\, p\right) + o(1).$$

5.2. The longest tube of a random walk.

Let $X_1, X_2, \ldots$ be i.i.d. integer valued random variables with common distribution $P(X_1 = i) = p_i$, $i \in \mathbb{Z}$; these are the steps of a random walk. Assume that for every $i \in \mathbb{Z}$ there exists a k_0 such that $P\left(X_1 + \cdots + X_k = i\right) > 0$ for $k \geq k_0$. For a fixed positive integer d define

$$B_k = \left\{(x_1, \ldots, x_k) \in \mathbb{Z}^k : \left|\sum_{t=i}^{j} x_t\right| < d,\ 1 \leq i \leq j \leq k\right\}.$$

A run of length k means that there exists an interval of d integers which contains the position of the random walk at each of k consecutive steps. In [5] the following approximation of $p(k)$ is derived. Let Q_d be the $d \times d$ matrix with entries p_{i-j}, $1 \leq i \leq d$, $1 \leq j \leq d$. If d is large enough, say $d \geq d_0$, then there exists a k such that Q_d^k is a positive matrix. Let ϱ_d denote the maximal characteristic value of Q_d, then $p(k) \sim c_d \varrho_d^k$ as $k \to \infty$ with a suitable multiplier c_d (see [5] for the exact value), thus (4.2) is fulfilled. For instance, if X_1 is symmetrically distributed, and $|X_1| \leq 1$, let $p_1 = p_{-1} = p$, $p_0 = q = 1 - 2p$; then

$$\varrho_d = q + p \cos\frac{\pi}{d+1}, \quad c_d = \frac{2}{d+1}\left(\cot\frac{\pi}{2(d+1)}\right)^2.$$

If both $X_{1,k}$ and $X_{j+1,k-j+1}$ remain within an interval of length d, but $X_{1,k+1}$ does not, then the two intervals cannot coincide, thus $X_{j+1,k-j}$ falls into a shorter interval. The probability of that is approximately $c_{d-1}\varrho_{d-1}^{k-j}$. Since $\varrho_{d-1} < \varrho_d$, this probability vanishes compared to $p(k+1)$, as $k \to \infty$. Hence (4.3) follows; and Theorems 4.1 and 4.2 can be applied with $p = \varrho_d$. In addition, by the periodicity of f_m we can write

$$P\left(M_n = m\right) = f_m\left(nc_d,\, \varrho_d\right) + o(1).$$

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6. Multivariate extensions

In this section multivariate extensions of Lemma 2.2 and Theorem 4.1 will be presented. First, let us consider the case of arrays with i.i.d. random variables within rows. Suppose property SH holds. Let us define recursively

$$Z_1(n) = W_n, \quad Z_j(n) = \max\{Y_{i,n} : 1 \le i \le n,\ Y_{i,n} < Z_{j-1}(n)\}\,,\ j > 1.$$

These random variables list the different values of the sample in decreasing order. Let $\mu_j(n)$ denote the multiplicity of $Z_j(n)$, that is,

$$\mu_j(n) = \#\{i \le N_n : Y_{i,n} = Z_j(n)\}\,.$$

Finally, let us introduce the gaps $\sigma_j(n) = Z_{j-1}(n) - Z_j(n)$.

The following limit theorem can be derived along the same lines as Lemma 2.2 was proved.

Theorem 6.1. *With the notations of Section 2 we have*

$$P\left(\mu_1(n) = m_1,\ \ldots,\ \mu_j(n) = m_j,\ \sigma_1(n) = s_1,\ \ldots,\ \sigma_j(n) = s_j\right) =$$
$$= \frac{m!}{m_1!\cdots m_j!}\prod_{i=1}^{j}(1-r)^{s_i(m_1+\cdots+m_i)}\left[(1-r)^{-m} - 1\right] f_m(c(n), 1-r) + o(1),$$

where $m = m_1 + \cdots + m_j$ *and* $n \to \infty$.

Corollary 6.1. *The joint asymptotic distribution of* $\mu_1(n), \ldots, \mu_j(n)$ *is the following.*

$$P\left(\mu_1(n) = m_1,\ \ldots,\ \mu_j(n) = m_j\right) =$$
$$= \frac{m!}{m_1!\cdots m_j!}\prod_{i=1}^{j-1}\frac{(1-r)^{m_1+\cdots+m_i}}{1-(1-r)^{m_1+\cdots+m_i}}\, f_m(c(n), 1-r) + o(1).$$

Given $\mu_1(n), \ldots, \mu_j(n)$*, the gaps* $\sigma_1(n), \ldots, \sigma_j(n)$ *are conditionally asymptotically independent and geometrically distributed, namely,* $\sigma_i(n)$ *with parameter*

$$1 - (1-r)^{\mu_1(n)+\cdots+\mu_i(n)}.$$

Remark 6.1. From the first part of Corollary 5.1 it is clear that the asymptotic joint distribution of $\mu_1(n), \ldots, \mu_j(n)$ given their sum $\mu_1(n) + \cdots + \mu_j(n)$ does not suffer from logarithmic periodicity; in fact, $\mu_1(n), \ldots, \mu_j(n)$ have a limiting conditional distribution.

Remark 6.2. If the samples are drawn from geometric distribution of parameter q, then $r = p$, $1 - r = q$ and in both of Theorem 5.1 and the first part of Corollary 5.1 $f_m(c(n), 1-r)$ can be replaced with $f_m(N_n, q)$. Besides, the second part of Corollary 5.1 is valid not only asymptotically, but also for every finite n, provided $N_n > \mu_1(n) + \cdots + \mu_j(n)$.

Finally, let us turn to the waiting times defined in Section 4. Suppose (4.2) and (4.3) hold. Define $\mathcal{A}_n = \{(i,k) : 1 \le i \le n-k,\ X_{i,k} \in B_{i,k}\}$, and let

$$\begin{aligned} Z_1(n) &= \max\{k : (i,k) \in \mathcal{A}_n\}, \\ Z_j(n) &= \max\left\{k : (i,k) \in \mathcal{A}_n, k < Z_{j-1}(n)\right\},\ j > 1. \end{aligned}$$

Let $M_j(n)$ denote the multiplicity of $Z_j(n)$, that is,

$$M_j(n) = \#\left\{(i,k) \in \mathcal{A}_n : k = Z_j(n)\right\},$$

and let $S_j(n) = Z_{j-1}(n) - Z_j(n)$. Then the following generalization of Theorem 4.1 can be proved as well.

Theorem 6.2. *Let* $\lambda(n) = \min\{k : E(k) \ge n\}$. *Then*

$$\begin{aligned} &P\left(M_1(n) = m_1,\, \dots,\, M_j(n) = m_j,\, S_1(n) = s_1,\, \dots,\, S_j(n) = s_j\right) = \\ &\qquad = \frac{m!}{m_1! \cdots m_j!} \prod_{i=1}^{j} p^{s_i(m_1 + \cdots + m_i)} \left[p^{-m} - 1\right] f_m\Big(\frac{n}{E(\lambda(n))}, p\Big) + o(1), \end{aligned}$$

where $m = m_1 + \cdots + m_j$ *and* $n \to \infty$.

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